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R-Deformed Heisenberg Algebra, Quantum Mechanics, and Virasoro Algebra

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We review the R-deformed Heisenberg algebra and its Fock space representation. We construct the R-deformed quantum mechanics in *N* dimensions, and propose a new R-deformed Virasoro algebra.

1. INTRODUCTION

Quantum groups or deformed Lie algebras imply specific deformations of classical Lie algebras. From a mathematical point of view, they are noncommutative associative Hopf algebras. The structure and representation theory of such algebras have been developed extensively by Jimbo (1985, 1986) and Drinfel'd (1986).

The R-deformed Heisenberg algebra, which is the deformation involving the reflection operator R($R^2 = 1$), was introduced by Vasiliev (1989, 1991) in the context of the higher spin algebras, and modified by other authors (Brink *et al.*, 1992, 1993) in the investigation of the quantum mechanical *N*-body Calogero model, which is related to the $(1 + 1)$-dimensional anyon (Leinaas and Myrhein, 1988).

The paper is arranged as follows: In Section 2, we review the R-deformed Heisenberg algebra (Pyuschay, 1996a,b; Filippov *et al.*, 1992) and its representation. In Section 3, we use these results to construct the R-deformed quantum mechanics in *N* dimensions. In Section 4, we present a new deformed Virasoro algebra which we call the R-deformed Virasoro algebra.

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2. REVIEW OF THE R-DEFORMED HEISENBERG ALGEBRA

In brief, the R-deformed Heisenberg algebra is generated by the generators a^+, a^-, 1 and the reflection operator R satisfying the following (anti) commutation relations:

$$[a^-, a^+] = 1 + \nu R; \qquad R^2 = 1 \tag{1}$$

$$Ra^+ + a^+R = Ra^- + a^-R = 0, \qquad [a^-, 1] = [a^+, 1] = [R, 1] = 0$$

where ν is a real deformation parameter. The reflection operator R is Hermitian and a^+ (a^-) plays the role of creation (annihilation) operator.

Let us introduce the Fock space basis $|m\rangle = C_m|0\rangle$, where $|0\rangle$ is the ground state, which is the vacuum state satisfying $a^-|0\rangle = 0$, $\langle 0|0\rangle = 1$, and $R|0\rangle = r|0\rangle$, where $r = \pm 1$, and the C_m are normalization constants. Then from the relation

$$[a^-, (a^+)^n] = (n + \tfrac{1}{2}(1 - (-1)^n \nu R)(a^+)^{n-1} \tag{2}$$

we get the action of the operator a^+a^- on the state $(a^+)^n|0\rangle$, $n = 0, 1, 2, \ldots,$

$$a^+a^-(a^+)^n|0\rangle = [n]_\nu (a^+)^n|0\rangle \tag{3}$$

where the ν-symbol is given by

$$[n]_\nu = n + \frac{\nu}{2}(1 + (-1)^{n+1}) \tag{4}$$

Here r is taken to be $+1$.

Hence, we conclude that in the case when $\nu > -1$, the space of representation (1) is infinite and given by the complete set of normalized vectors

$$|m\rangle = \frac{(a^+)^m}{\sqrt{[m]_\nu!}}|0\rangle, \qquad \langle m|m'\rangle = \delta_{mm'} \tag{5}$$

where $[m]_\nu! = [m]_\nu \ldots [1]_\nu$.

In what follows, it is convenient to introduce the operators

$$(\pi_\pm) = \tfrac{1}{2}(1 \pm R) \tag{6}$$

which satisfy the equalities $(\pi_\pm)^2 = \pi_\pm$, $\pi_+\pi_- = 0$, and $\pi_+ + \pi_- = 1$, and the number operators **N** with

$$a^+a^- = \mathbf{N} + \nu\pi_- \tag{7}$$

where **N** satisfies the commutation relations

$$[a^-, \mathbf{N}] = a^-, \qquad [a^+, \mathbf{N}] = -a^+ \tag{8}$$

and $\mathbf{N}|0\rangle = 0$. As a consequence of the above equality, we get

$$\mathbf{N}|m\rangle = m|m\rangle \tag{9}$$

Using Eqs. (7) and (1), one obtains

$$a^{-}a^{+} = \mathbf{N} + 1 + \nu\pi_{+} \tag{10}$$

Combining Eqs. (10) and (7), we derive the following expression for the number operator:

$$\mathbf{N} = \tfrac{1}{2}\{a^{-}, a^{+}\} - \tfrac{1}{2}(1 + \nu) \tag{11}$$

One can realize the R operator in terms of the operators $a^{\pm}$ by means of Eq.(11):

$$R = \cos(\pi\mathbf{N}) \tag{12}$$

Then, the R operator acts in the state $|m\rangle$ as

$$R|m\rangle = \cos(\pi\mathbf{N})|m\rangle = (-1)^{m}|m\rangle \tag{13}$$

This equality explains the name of the operator R, namely, the reflection operator. So we can rewrite the (11) for the number operator in the following form:

$$\mathbf{N} = a^{+}a^{-} + \frac{\nu}{2}(\cos \pi\mathbf{N} - 1) \tag{14}$$

Here the R-deformed oscillator algebra (1) can be reduced to the compact form

$$a^{-}a^{+} = f(a^{+}a^{-}) \tag{15}$$

where $f(a^{+}a^{-}) = a^{+}a^{-} + \nu(\cos \pi\mathbf{N}a^{+}a^{-})$.

To conclude this section, we note that one gets the realization of the R-deformed Heisenberg algebra in terms of the undeformed oscillator algebra generated by (b^{-}, b^{+}) where a^{+} and a^{-} are expressed as

$$a^{-} = G(\mathbf{N}_{b})b^{-}, \qquad a^{+} = b^{+}G(\mathbf{N}_{b}) \tag{16}$$

G is a Hermitian function of the number operator $\mathbf{N}_{b} = b^{+}b^{-}$,

$$G(\mathbf{N}_{b}) = \sqrt{1 + \frac{\nu}{2(\mathbf{N}_{b} + 1)}(1 + (-1)^{\mathbf{N}_{b}})} \tag{17}$$

where $\nu > -1$.

If we put $\nu = -(2p + 1)$, $p = 1, 2, 3, \ldots$, the function $G(\mathbf{N}_{b})$ takes the value zero and from (2), we get $(a^{+})^{2p+1} = (a^{-})^{2p+1} = 0$. This leads us to discuss the para-Grassman representation of the R-deformed Heisenberg algebra. In this representation a^{+} can be intepreted as a para-Grassman variable

θ, $(\theta)^{2p+1} = 0$, and the annihilation operator a^- as a differentiation ∂_θ, $(\partial_\theta)^{2p+1} = 0$, which satisfy (Filippov *et al.*, 1992)

$$\begin{aligned} [\theta, \partial_\theta] &= 1 - (2p + 1)R \\ \{\partial_\theta, R\} &= \{\theta, R\} = 0 \end{aligned} \tag{18}$$

So the R-deformed Heisenberg algebra is nonthing but the para-Grassman algebra of order $2p + 1$.

In this realization, thanks to the nilpotency condition $(a^+)^{2p+1}$, the Fock space in finite and its basis is given by

$$F = |m\rangle, \qquad m = 1, 2, 3, \ldots, 2p \tag{19}$$

3. R-DEFORMED QUANTUM MECHANICS IN *N* DIMENSIONS

In this section, we construct the R-deformed harmonic oscillator in one dimension; the generalization to N dimensions is staightforward. In order to formulate it, we define the position and momentum operators as

$$\begin{aligned} X &= \frac{1}{\sqrt{2}}(a^- + a^+) \\ P_\nu &= \frac{i}{\sqrt{2}}(a^- - a^+) \end{aligned} \tag{20}$$

Then the Hamiltonian of the R-deformed harmonic oscillator is given by

$$H_\nu = \tfrac{1}{2}(P_\nu^2 + X^2) = \tfrac{1}{2}\{a^+, a^-\} \tag{21}$$

where $a^\pm = (1/\sqrt{2})(X \pm iP_\nu)$ and P_ν is the deformed momentum operator $P_\nu = -i(d/dX - \nu R/2X)$.

The R-deformed cannonical relation can be expressed as

$$XP_\nu - P_\nu X = i(1 + \nu R) \tag{22}$$

Now, we look at the energy spectrum of such systems, which is given by

$$H_\nu|m\rangle = E_\nu(m)|m\rangle \tag{23}$$

where $E_\nu(m) = ([m]_\nu + \frac{1}{2} - (-1)^{m+1}\nu)$.

For the N-dimensional case, the Hamiltonian of R-deformed harmonic oscillator in N dimensions is given by

$$H_\nu = (H_\nu)_1 + (H_\nu)_2 + \ldots (H_\nu)_N \tag{24}$$

and the energy spectrum is obtained by the following sum:

$$E_\nu(m_1, m_2, \ldots m_N) = \Bigg([m_1]_\nu + [m_2]_\nu \ldots [m_N]_\nu + \frac{N}{2} - \{(-1)^{m_1+1} + \ldots (-1)^{m_N+1}\}\nu\Bigg) \qquad (25)$$

The state $|m_1, m_2, \ldots, m_N\rangle$ is obtained by applying the creation operators to the ground state $|0, 0, 0, \ldots\rangle$,

$$|m_1, \ldots, m_N\rangle = \frac{(a_1^+)^{m_1} \ldots (a_N^+)^{m_N}}{\sqrt{[m_1]_\nu!} \ldots \sqrt{[m_N]_\nu!}}\,|0, 0, \ldots\rangle \qquad (26)$$

4. A NEW R-DEFORMED VIRASORO ALGEBRA

In this section, show that the R-deformed oscillator algebra may be used to construct a new deformed Virasoro algebra, which we call R-deformed Virasoro algebra. To do this, we adopt the approach for the undeformed case (Chichian *et al.*, 1990), where the generators L_m ($m \in Z$) are expressed as $L_m = (b^+)^{m+1}b^2$, where $b^\pm$ satisfy the undeformed oscillator algebra $[b^+, b^-] = 1$. Then the generators L_m ($m \in Z$) obey the following commutation relations:

$$[L_m, L_n] = (m - n)L_{m+n} \qquad (27)$$

At this point, we recall that the q-deformation of this algebra was introduced by Cutright and Zachos (1990) and investigated on many occasions (EL Kinani and Zakkari, 1995; Devchand and Saveliev, 1991; Aizawa and Sato, 1991):

$$[L_m, L_n](q^{m-n}, q^{n-m}) = [m - n]_q L_{m+n} \qquad (28)$$

where $[A, B]_{(p,q)} = pAB - qBA$ and $[x]_q = (g^x - g^{-x})/(q - q^{-1})$.

Now, we turn to the R-deformed case; let us introduce the generators $L_m(\nu)$ as

$$L_m(\nu) = (a^+)^{m+1}a^- \qquad (29)$$

Thanks to Eq. (2), we get the following commutation relations for $L_m(\nu)$:

$$[L_m(\nu), L_n(\nu)] = ((m - n) + \tfrac{1}{2}((-1)^n - (-1)^m\nu)R)L_{m+n}(\nu) \qquad (30)$$

which goes to the ordinary Virasoro algebra for $\nu \to 0$. However, when $\nu \neq 0$, it is something new. One can ask about the commutation between R and the generators L_m ($m \in Z$); from Eq. (1), one easily finds $[R, L_m] = \alpha(m)RL_m$, where $\alpha(m) = 1 - (-1)^m$. For m even, $\alpha(m) = 0$, and R commutes with L_m; for m odd, $\alpha(m) = 2$, and we have $[R, L_m] = 2RL_m$.

5. CONCLUDING REMARKS

In this paper, we used the R-deformed Heisenberg algebra to construct R-deformed quantum mechanics in N dimensions. Morever we proposed new deformation of Virasoro algebra which we call the R-deformed Virasoro algebra. Finally, we note that one can construct in same way the R-deformed W_∞-algebra; further details on this and the connection between R-deformation and q-deformation are given elsewhere (EL Kinani, 1999).

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